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SU(3) in an SU(2) basis: an alternative to hypercharge

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Abstract. An analysis of representations of SU(3) in an SU(2) basis is given using SU(2) shift operators. It is found that, apart from the hypercharge operator, there are two further SU(2) scalar Hermitian operators whose diagonalization leads to new bases for the representation space. These are modified to provide a basis which appears more suited to a description of the pseudoscalar meson octet than does the usual basis with diagonal hypercharge.

1. Introduction

In a previous paper (Hughes and Yadegar 1976) the algebra consisting of the generators $l_0, l_\pm$ of O(3) and two mutually Hermitian pairs of two-dimensional O(3) tensor operators, $q_{\pm\frac{1}{2}}$ and $\bar{q}_{\pm\frac{1}{2}}$, was considered. It was shown that out of these operators two pairs of operators could be constructed which shift l and m by $\pm\frac{1}{2}$, $l(l+1)$ and m being the eigenvalues of the O(3) Casimir, and l_0, respectively. These operators were used to give a complete classification and analysis of representations of the Lie algebra of the seven-dimensional non-compact group $O(3)_\wedge(T_2\times\bar{T}_2)$ formed by requiring that the $q_{\pm\frac{1}{2}}$ and $\bar{q}_{\pm\frac{1}{2}}$ mutually commute. It was found that the enveloping algebra contained three O(3) scalar operators $Y_\pm$ and Y_0, which themselves generated, on normalization with a suitable Casimir-dependent term, an O(3) group.

The Lie algebra of SU(3) also contains the operators l_0, $l_\pm$, $q_{\pm\frac{1}{2}}$ and $\bar{q}_{\pm\frac{1}{2}}$, together with an O(3) scalar operator p_0. The subgroup generated by l_0, $l_\pm$ is usually referred to as an SU(2) subgroup, since in the decomposition of its representations with respect to this subgroup half-integral l and m values occur. This also serves to distinguish it from the O(3) subgroup considered by, for instance, Racah (1962), Hughes (1973a, b), for which only integral values of l and m occur. In the octet model for hadrons, SU(2) may be, for instance, the isotopic spin subgroup, in which case p_0 is the hypercharge operator.

For SU(3) the $q_{\pm\frac{1}{2}}$ and $\bar{q}_{\pm\frac{1}{2}}$ no longer commute, but their commutation relations with l_o, $l_\pm$ are precisely the same as for $O(3)_\wedge(T_2\times\bar{T}_2)$ and hence precisely the same l and m shift operators may be constructed. Moreover, its enveloping algebra contains precisely the same SU(3) scalar operators $Y_\pm$ and Y_0, although they no longer generate an O(3) group. The operators Y_0, (Y_++Y_-) and $\mathrm{i}(Y_+-Y_-)$ are all Hermitian and any one of these, together with l_0 and the SU(2) Casimir L^2, forms a complete set of commuting operators with respect to the representations of SU(3). Y_0 can be expressed in terms of L^2, p_0 and the third-order invariant I_3, so its diagonalization is completely equivalent to the diagonalization of p_0. This leads to the usual classification of representations of

SU(3) in terms of weights so familiar in the octet model for hadrons (see for instance de Swart 1963). The existence of $(Y_+ + Y_-)$ and $\mathrm{i}(Y_+ - Y_-)$, and the fact that the diagonalization of either of these is an alternative to the diagonalization of p_0, does not appear to have been realized before.

We therefore give here an analysis of the irreducible representations of SU(3) in an SU(2) basis using the shift operators $A^{\pm\frac{1}{2}}$, $\bar{A}^{\pm\frac{1}{2}}$. It will be seen that the familiar basis in which p_0 is diagonal corresponds to the diagonalization of products of shift operators of the type $A^{\pm\frac{1}{2}}\bar{A}^{\mp\frac{1}{2}}$ or $\bar{A}^{\pm\frac{1}{2}}A^{\mp\frac{1}{2}}$, whereas the unfamiliar choice of basis in which $(Y_+ + Y_-)$ or $\mathrm{i}(Y_+ - Y_-)$ is diagonal corresponds to the diagonalization of $(A^{\pm\frac{1}{2}}A^{\mp\frac{1}{2}} - \bar{A}^{\pm\frac{1}{2}}\bar{A}^{\mp\frac{1}{2}})$ or $\mathrm{i}(A^{\pm\frac{1}{2}}A^{\mp\frac{1}{2}} + \bar{A}^{\pm\frac{1}{2}}\bar{A}^{\mp\frac{1}{2}})$, respectively.

Although we do not claim any physical interpretation of the operators $(Y_+ + Y_-)$, $\mathrm{i}(Y_+ - Y_-)$, we consider the pseudoscalar meson octet and show that whereas the mesons K^0 and $\bar{\mathrm{K}}^0$ are eigenvectors of hypercharge p_0, the more physical short- and long-lived mesons $\mathrm{K}^0_{\mathrm{S}}$ and $\mathrm{K}^0_{\mathrm{L}}$, together with the other mesons of the octet, are eigenstates of a suitable linear combination of $(Y_+l_+ + Y_-l_-)$ and $\mathrm{i}(Y_+l_+ - Y_-l_-)$ if, and only if, CPT conservation is assumed to hold.

We summarize the SU(3) commutation relations and properties of the shift operators in § 2 and in § 3 give a complete classification of matrix elements for the general irreducible representations (p, q) for the case of p_0 diagonal. We do not give a general treatment for the cases in which $(Y_+ + Y_-)$ and $\mathrm{i}(Y_+ - Y_-)$ are diagonal, merely giving in § 4 their eigenvalues and eigenvectors for the eight-dimensional representation (2, 1) and the 27-dimensional representation (4, 2). The pseudoscalar meson octet is discussed in § 5.

2. SU(3) and its SU(2) shift operators

The usual choice of basis for the Lie algebra of SU(3) consists of the basis (H_1, H_2) of the Cartan subalgebra and the root vectors $E_{\pm\alpha}, E_{\pm\beta}, E_{\pm\bar{\beta}}$, whose commutation relations are given by, for instance, Baird and Biedenharn (1963). In order to make the notation correspond more precisely to that of Hughes and Yadegar (1976) we define

$$\begin{aligned} &l_0 = \sqrt{3}H_1, \qquad && l_\pm = \sqrt{6}E_{\pm\alpha}, \qquad && p_0 = 2H_2, \\ &q_{\frac{1}{2}} = \sqrt{6}E_{\bar{\beta}}, && q_{-\frac{1}{2}} = -\sqrt{6}E_{-\beta}, && \bar{q}_{-\frac{1}{2}} = \sqrt{6}E_{\beta}, \qquad && \bar{q}_{\frac{1}{2}} = \sqrt{6}E_{-\bar{\beta}}. \end{aligned} \tag{1}$$

Their non-vanishing commutation relations are

$$\begin{aligned} &[l_0, l_\pm] = \pm l_\pm, && [l_+, l_-] = 2l_0, \\ &[l_0, \overset{(-)}{q}_{\pm\frac{1}{2}}] = \pm\tfrac{1}{2}\overset{(-)}{q}_{\pm\frac{1}{2}}, && [l_\pm, \overset{(-)}{q}_{\mp\frac{1}{2}}] = \overset{(-)}{q}_{\pm\frac{1}{2}}, \\ &[q_{\pm\frac{1}{2}}, \bar{q}_{\mp\frac{1}{2}}] = l_0 \mp \tfrac{3}{2}p_0, && [q_{\pm\frac{1}{2}}, \bar{q}_{\pm\frac{1}{2}}] = \mp l_\pm, \\ &[p_0, q_{\pm\frac{1}{2}}] = -q_{\pm\frac{1}{2}}, && [p_0, \bar{q}_{\pm\frac{1}{2}}] = \bar{q}_{\pm\frac{1}{2}}, \end{aligned} \tag{2}$$

and they satisfy the Hermiticity conditions

$$l_0^\dagger = l_0, \qquad l_\pm^\dagger = l_\mp, \qquad p_0^\dagger = p_0, \qquad q_{\pm\frac{1}{2}}^\dagger = \pm\bar{q}_{\mp\frac{1}{2}}. \tag{3}$$

The invariants I_2, I_3, defined by Baird and Biedenharn (1963) are given by

$$12I_2 = 4(L^2 + q_{\frac{1}{2}}\bar{q}_{-\frac{1}{2}} - q_{-\frac{1}{2}}\bar{q}_{\frac{1}{2}}) + 3p_0(p_0 + 2) \tag{4}$$

and

$$48I_3 = (p_0^2 - 4 - 12I_2 + 12L^2)p_0 - 8l_0(q_{-\frac{1}{2}}\bar{q}_{\frac{1}{2}} + q_{\frac{1}{2}}\bar{q}_{-\frac{1}{2}}) + 8(l_- q_{\frac{1}{2}}\bar{q}_{-\frac{1}{2}} - l_+ q_{-\frac{1}{2}}\bar{q}_{-\frac{1}{2}}) + 8L^2. \tag{5}$$

These are both Hermitian.

The SU(2) scalar operators of Hughes and Yadegar (1976) are unchanged:

$$Y_+ = -(2q_{\frac{1}{2}}q_{-\frac{1}{2}}l_0 + q_{-\frac{1}{2}}q_{-\frac{1}{2}}l_+ - q_{\frac{1}{2}}q_{\frac{1}{2}}l_-) \tag{6}$$

$$Y_0 = q_{\frac{1}{2}}\bar{q}_{-\frac{1}{2}}l_0 + q_{-\frac{1}{2}}\bar{q}_{\frac{1}{2}}l_0 - q_{\frac{1}{2}}\bar{q}_{\frac{1}{2}}l_- + q_{-\frac{1}{2}}\bar{q}_{-\frac{1}{2}}l_+ \tag{7}$$

$$Y_- = 2\bar{q}_{\frac{1}{2}}\bar{q}_{-\frac{1}{2}}l_0 + \bar{q}_{-\frac{1}{2}}\bar{q}_{-\frac{1}{2}}l_+ - \bar{q}_{\frac{1}{2}}\bar{q}_{\frac{1}{2}}l_- \tag{8}$$

and they satisfy $Y_+^\dagger = Y_-$, $Y_0^\dagger = Y_0$. Hence Y_0, $(Y_+ + Y_-)$ and $\mathrm{i}(Y_+ - Y_-)$ are Hermitian operators. Unlike the case of $\mathrm{O}(3)_\mathrm{A}(\mathrm{T}_2 \times \bar{\mathrm{T}}_2)$, however, they do not satisfy the commutation relations for an O(3) group; for instance

$$[Y_+, Y_-] = \tfrac{3}{2}(Y_2 - L^2(3p_0 + 1))(4I_2 - 4L^2 - p_0^2) - 3L^2p_0(4L^2 + 1). \tag{9}$$

Y_0 is in fact related to I_3 by

$$48I_3 = p_0(p_0^2 - 4 - 12I_2 + 12L^2) + 8L^2 - 8Y_0 \tag{10}$$

so the diagonalization of Y_0 is equivalent to that of p_0. We shall therefore consider Y_0 no further in this paper. Y_+ and Y_- satisfy

$$[p_0, Y_+] = -2Y_+, \qquad [p_0, Y_-] = 2Y_- \tag{11}$$

and so act as shift operators for eigenvectors of p_0.

The l and m shift operators are given by

$$\overset{(-)}{O}{}^{\frac{1}{2},\frac{1}{2}} = \overset{(-)}{q}_{\frac{1}{2}}(R + l_0 + 1) + \overset{(-)}{q}_{-\frac{1}{2}}l_+ \tag{12}$$

$$\overset{(-)}{O}{}^{-\frac{1}{2},-\frac{1}{2}} = -\overset{(-)}{q}_{-\frac{1}{2}}(R + l_0) + \overset{(-)}{q}_{\frac{1}{2}}l_- \tag{13}$$

where R is the operator whose eigenvalue is l. Provided they act to the right on eigenvectors of R and l_0, these operators may be replaced by their eigenvalues. The fact that $\overset{(-)}{O}{}^{\pm\frac{1}{2},\pm\frac{1}{2}}$ act as l and m shift operators follows from the commutation relations

$$[L^2, \overset{(-)}{O}{}^{\frac{1}{2},\frac{1}{2}}] = \overset{(-)}{O}{}^{\frac{1}{2},\frac{1}{2}}(R + \tfrac{3}{4}), \qquad [L^2, \overset{(-)}{O}{}^{-\frac{1}{2},-\frac{1}{2}}] = -\overset{(-)}{O}{}^{-\frac{1}{2},-\frac{1}{2}}(R + \tfrac{1}{4})$$

$$\text{and} \qquad [l_0, \overset{(-)}{O}{}^{\pm\frac{1}{2},\pm\frac{1}{2}}] = \pm\tfrac{1}{2}\overset{(-)}{O}{}^{\pm\frac{1}{2},\pm\frac{1}{2}}.$$

We shall find it more convenient to use the normalized shift operators

$$\overset{(-)}{A}{}_l^{\frac{1}{2}} = (l + m + 1)^{-\frac{1}{2}}\overset{(-)}{O}{}_{l,m}^{\frac{1}{2},\frac{1}{2}}, \qquad \overset{(-)}{A}{}_l^{-\frac{1}{2}} = (l + m)^{-\frac{1}{2}}\overset{(-)}{O}{}_{l,m}^{-\frac{1}{2},-\frac{1}{2}}. \tag{14}$$

This effectively means that the internal SU(2) structure of the SU(3) representations has been divided out.

The Hermiticity properties of these shift operators are worked out by Hughes and Yadegar (1976) and are

$$\langle\gamma', l \pm \tfrac{1}{2}|A_l^{\pm\frac{1}{2}}|\gamma, l\rangle = \mp\frac{(2l+1)}{2(l + \frac{1}{2} \pm \frac{1}{2})}\langle\gamma', l \pm \tfrac{1}{2}|(\bar{A}_{l\pm\frac{1}{2}}^{\mp\frac{1}{2}})^\dagger|\gamma, l\rangle \tag{15}$$

where γ denotes all additional labels, and the m dependence of the states has been suppressed. From equation (15) many useful relations between the matrix elements of the $\overset{(-)}{A}{}^{\pm\frac{1}{2}}$ can be derived. We shall need the following in the next section:

$$\sum_{\gamma'} |\langle\gamma', l\pm\tfrac{1}{2}|A_l^{\pm\frac{1}{2}}|\gamma, l\rangle|^2 = \mp\frac{(2l+1)}{2(l+\frac{1}{2}\pm\frac{1}{2})}\langle\gamma, l|\bar{A}_{l\pm\frac{1}{2}}^{\mp\frac{1}{2}}A_l^{\pm\frac{1}{2}}|\gamma, l\rangle \tag{16}$$

$$\sum_{\gamma'} |\langle\gamma', l\pm\tfrac{1}{2}|\bar{A}_l^{\pm\frac{1}{2}}|\gamma, l\rangle|^2 = \pm\frac{(2l+1)}{2(l+\frac{1}{2}\pm\frac{1}{2})}\langle\gamma, l|A_{l\pm\frac{1}{2}}^{\mp\frac{1}{2}}\bar{A}_l^{\pm\frac{1}{2}}|\gamma, l\rangle. \tag{17}$$

The expression for l-commuting products of the shift operators do differ from those of $O(3)_\Lambda(T_2\times\bar{T}_2)$. They are given by the following equations, where we assume they act to the right on states $|\gamma, l\rangle$:

$$A_{l-\frac{1}{2}}^{\frac{1}{2}}A_l^{-\frac{1}{2}} = A_{l+\frac{1}{2}}^{-\frac{1}{2}}A_l^{\frac{1}{2}} = Y_+ \tag{18}$$

$$\bar{A}_{l-\frac{1}{2}}^{\frac{1}{2}}\bar{A}_l^{-\frac{1}{2}} = \bar{A}_{l+\frac{1}{2}}^{-\frac{1}{2}}\bar{A}_l^{\frac{1}{2}} = -Y_- \tag{19}$$

$$A_{l-\frac{1}{2}}^{\frac{1}{2}}\bar{A}_l^{-\frac{1}{2}} = 6I_3-3I_2l+l(l+1)(l-1)-\tfrac{1}{8}p_0(12l^2-6lp_0+p_0^2-12I_2-4) \tag{20}$$

$$A_{l+\frac{1}{2}}^{-\frac{1}{2}}\bar{A}_l^{\frac{1}{2}} = 6I_3+3I_2(l+1)-l(l+1)(l+2)-\tfrac{1}{8}p_0(12l^2+24l+6lp_0+p_0^2+6p_0-12I_2+8) \tag{21}$$

$$\bar{A}_{l-\frac{1}{2}}^{\frac{1}{2}}A_l^{-\frac{1}{2}} = 6I_3+3I_2l-l(l+1)(l-1)-\tfrac{1}{8}p_0(12l^2+6lp_0+p_0^2-12I_2-4) \tag{22}$$

$$\bar{A}_{l+\frac{1}{2}}^{-\frac{1}{2}}A_l^{\frac{1}{2}} = 6I_3-3I_2(l+1)+l(l+1)(l+2) -\tfrac{1}{8}p_0(12l^2+24l-6lp_0+p_0^2-6p_0-12I_2+8). \tag{23}$$

These equations justify the remark made in the introduction that the diagonalization of p_0 is equivalent to that of operators $A^{\pm\frac{1}{2}}\bar{A}^{\mp\frac{1}{2}}$ and $\bar{A}^{\pm\frac{1}{2}}A^{\mp\frac{1}{2}}$, whereas the diagonalization of $(Y_+ + Y_-)$ or $i(Y_+ - Y_-)$ imply the diagonalization of $(A^{\pm\frac{1}{2}}A^{\mp\frac{1}{2}} - \bar{A}^{\pm\frac{1}{2}}\bar{A}^{\mp\frac{1}{2}})$ and $i(A^{\pm\frac{1}{2}}A^{\mp\frac{1}{2}} + \bar{A}^{\pm\frac{1}{2}}\bar{A}^{\mp\frac{1}{2}})$, respectively.

Finally, from equations (20) to (23) we may derive the following relations which will be needed in the analysis of the next section:

$$\begin{aligned} p_0\{(\bar{A}_{l+\frac{1}{2}}^{-\frac{1}{2}}A_l^{\frac{1}{2}} - A_{l+\frac{1}{2}}^{-\frac{1}{2}}A_l^{\frac{1}{2}}) + 4(l+1)[(2l+1)(2l+3)-3I_2]\} \\ = 72(l+1)I_3 - 6(l+1)(A_{l+\frac{1}{2}}^{-\frac{1}{2}}\bar{A}_l^{\frac{1}{2}} + \bar{A}_{l+\frac{1}{2}}^{-\frac{1}{2}}A_l^{\frac{1}{2}}) \end{aligned} \tag{24}$$

$$p_0^2 = \frac{2}{3(l+1)}(\bar{A}_{l+\frac{1}{2}}^{-\frac{1}{2}}A_l^{\frac{1}{2}} - A_{l+\frac{1}{2}}^{-\frac{1}{2}}\bar{A}_l^{\frac{1}{2}}) + 4I_2 - \tfrac{4}{3}l(l+2) \tag{25}$$

$$A_{l-\frac{1}{2}}^{\frac{1}{2}}\bar{A}_l^{-\frac{1}{2}} = \frac{(2l+1)}{2(l+1)}\bar{A}_{l+\frac{1}{2}}^{-\frac{1}{2}}A_l^{\frac{1}{2}} + \frac{1}{2(l+1)}A_{l+\frac{1}{2}}^{-\frac{1}{2}}\bar{A}_l^{\frac{1}{2}} + \tfrac{1}{2}(3p_0-2l)(2l+1) \tag{26}$$

$$\bar{A}_{l-\frac{1}{2}}^{\frac{1}{2}}A_l^{-\frac{1}{2}} = \frac{1}{2(l+1)}\bar{A}_{l+\frac{1}{2}}^{-\frac{1}{2}}A_l^{\frac{1}{2}} + \frac{(2l+1)}{2(l+1)}A_{l+\frac{1}{2}}^{-\frac{1}{2}}\bar{A}_l^{\frac{1}{2}} + \tfrac{1}{2}(3p_0+2l)(2l+1). \tag{27}$$

3. Analysis of the irreducible unitary representations of SU(3)

The classification of irreducible unitary representations (IUR) of SU(3) is very well known (see for instance de Swart 1963, Baird and Biedenharn 1963). They are uniquely specified by the pair (p, q), where p, q are non-negative integers such that $p \geq q$, have dimension $\frac{1}{2}(p-q+1)(p+2)(q+1)$ and correspond to the following values of the invariants:

$$I_2=\tfrac{1}{9}(p^2+q^2-pq+3p), \qquad I_3=\tfrac{1}{162}(p-2q)(2p+3-q)(p+q+3). \tag{28}$$

(p, q) and $(p, p-q)$ are mutually contragredient.

We give here the analysis in terms of the shift operators $\overset{(-)}{A}{}^{\pm}$ of (p, q) in the SU(2) basis. We shall denote the states by $|l, j\rangle$, suppressing the p, q and m labels, where j distinguishes between different states of the same l value and will be defined in such a way as to diagonalize $\bar{A}^{\pm}A^{\mp}$ and $A^{\pm}\bar{A}^{\mp}$ and therefore also p_0. We shall also give the actions of $Y_{\pm}$ on $|l, j\rangle$; once these are known the eigenvectors and eigenvalues of $(Y_+ + Y_-)$ and $\mathrm{i}(Y_+ - Y_-)$ can be obtained, although we shall do so only for the IUR (2, 1) and (4, 2) in the next section.

The first task is to determine the maximum l value, $\bar{l}$, which must be non-degenerate. This is determined by the requirement that

$$A^{\frac{1}{2}}|\bar{l}, 1\rangle=\bar{A}^{\frac{1}{2}}|\bar{l}, 1\rangle=0$$

which imply that

$$\bar{A}^{-\frac{1}{2}}A^{\frac{1}{2}}|\bar{l}, 1\rangle=A^{-\frac{1}{2}}\bar{A}^{\frac{1}{2}}|\bar{l}, 1\rangle=0.$$

Using equations (24) and (25) we see that p_0 is diagonal on $|\bar{l}, 1\rangle$ and its value satisfies

$$p_0^2=\tfrac{4}{3}[3I_2-\bar{l}(\bar{l}+2)]$$

and

$$p_0[(2\bar{l}+1)(2\bar{l}+3)-3I_2]=18I_3.$$

From these equations and the values of I_2, I_3 in terms of p and q one obtains, after some straightforward calculations, the following equation for $\bar{l}$:

$$(2\bar{l}-p)(2\bar{l}+p+4)(2\bar{l}-q+1)(2\bar{l}+q+3)(2\bar{l}-p+q+1)(2\bar{l}+p-q+3)=0.$$

The only possibilities consistent with $\bar{l}\geqslant 0$ are $\bar{l}=\frac{1}{2}p$, $\frac{1}{2}(q-1)$ and $\frac{1}{2}(p-q-1)$. The latter two give rise to negative values of the operator $\bar{A}^{\frac{1}{2}}_{\bar{l}-\frac{1}{2}}A^{-\frac{1}{2}}_{\bar{l}}$, which is seen from equation (16) to be positive definite, and can therefore be excluded. Hence we see that $\bar{l}=\frac{1}{2}p$.

Using the formulae given at the end of § 2, the following actions of the various operators may be obtained:

$$p_0|\tfrac{1}{2}p, 1\rangle=\tfrac{1}{3}(p-2q)|\tfrac{1}{2}p, 1\rangle \tag{29}$$

$$Y_+|\tfrac{1}{2}p, 1\rangle=A^{\frac{1}{2}}A^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle=0 \tag{30}$$

$$Y_-|\tfrac{1}{2}p, 1\rangle=-\bar{A}^{\frac{1}{2}}\bar{A}^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle=0 \tag{31}$$

$$\bar{A}^{\frac{1}{2}}A^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle=(p-q)(p+1)|\tfrac{1}{2}p, 1\rangle \tag{32}$$

$$A^{\frac{1}{2}}\bar{A}^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle=-q(p+1)|\tfrac{1}{2}p, 1\rangle. \tag{33}$$

We next consider the $l=(\frac{1}{2}p-\frac{1}{2})$ states; since we have two independent shift operators, $A^{-\frac{1}{2}}$ and $\bar{A}^{-\frac{1}{2}}$, there will in general be two independent states which we may choose to be normalized and such that

$$|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle\propto A^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle, \qquad |\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle\propto \bar{A}^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle.$$

From (30) and (31) we see that

$$A^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle = 0, \qquad \bar{A}^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle = 0$$

whereas (32) and (33) imply that

$$\bar{A}^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle \propto |\tfrac{1}{2}p, 1\rangle, \qquad A^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle \propto |\tfrac{1}{2}p, 1\rangle.$$

From the equations (16) and (17) one now obtains

$$|\langle\tfrac{1}{2}p-\tfrac{1}{2}, 1|A^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle|^2 = \frac{p+1}{p}\langle\tfrac{1}{2}p, 1|\bar{A}^{\frac{1}{2}}A^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle$$

$$|\langle\tfrac{1}{2}p-\tfrac{1}{2}, 2|\bar{A}^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle|^2 = -\frac{p+1}{p}\langle\tfrac{1}{2}p, 1|A^{\frac{1}{2}}\bar{A}^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle.$$

So far the states $|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle$ and $|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle$ are determined only up to a phase factor. The phases are now determined by the requirement that $\langle\tfrac{1}{2}p-\tfrac{1}{2}, 1|A^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle$ and $\langle\tfrac{1}{2}p-\tfrac{1}{2}, 2|\bar{A}^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle$ be real and non-negative. With this choice we now obtain

$$A^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle = (p+1)[(p-q)/p]^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle \tag{34}$$

and

$$\bar{A}^{-\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle = (p+1)(q/p)^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle. \tag{35}$$

These two equations, together with (32) and (33), then yield

$$\bar{A}^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle = [p(p-q)]^{\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle \tag{36}$$

$$A^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle = -[pq]^{\frac{1}{2}}|\tfrac{1}{2}p, 1\rangle \tag{37}$$

$$A^{-\frac{1}{2}}\bar{A}^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle = (p-q)(p+1)|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle \tag{38}$$

$$\bar{A}^{-\frac{1}{2}}A^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle = -q(p+1)|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle. \tag{39}$$

Also,

$$\bar{A}^{-\frac{1}{2}}A^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle = 0, \qquad A^{-\frac{1}{2}}\bar{A}^{\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle = 0. \tag{40}$$

We therefore see that $|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle$ and $|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle$ are eigenvectors corresponding to distinct eigenvalues of both of the Hermitian operators $\bar{A}^{-\frac{1}{2}}A^{\frac{1}{2}}$ and $A^{-\frac{1}{2}}\bar{A}^{\frac{1}{2}}$; this guarantees their orthogonality.

From equations (36)–(39), the actions of p_0, $Y_{\pm}$, $\bar{A}^{\frac{1}{2}}A^{-\frac{1}{2}}$ and $A^{\frac{1}{2}}\bar{A}^{-\frac{1}{2}}$ can be obtained with the help of equations (24) to (27). We are then able to define the $l=\tfrac{1}{2}p-1$ states by

$$|\tfrac{1}{2}p-1, 1\rangle \propto A^{-\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle, \qquad |\tfrac{1}{2}p-1, 3\rangle \propto \bar{A}^{-\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle$$

$$|\tfrac{1}{2}p-1, 2\rangle \propto \bar{A}^{-\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 1\rangle \propto A^{-\frac{1}{2}}|\tfrac{1}{2}p-\tfrac{1}{2}, 2\rangle.$$

This should suffice to illustrate the method of defining the various l states and the calculation of the actions of the various operators. We state the actions on the general $|l, j\rangle$ state in the form of a theorem which we prove by induction. The states of (p, q) and the actions of the shift operators are depicted in figure 1 for the case $q \leqslant \tfrac{1}{2}p$.

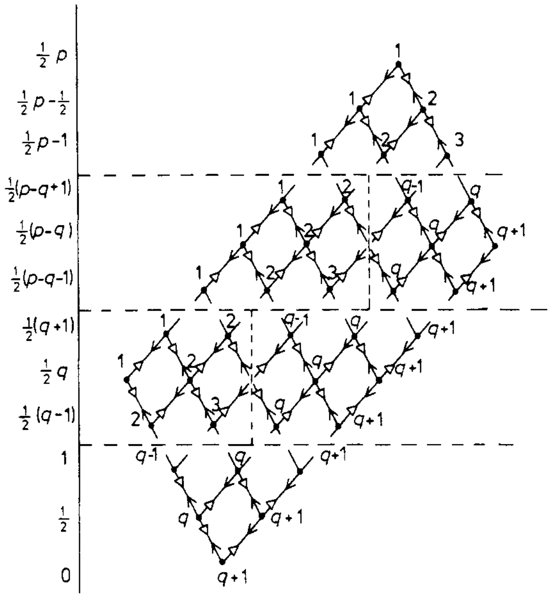

Figure 1. States of the IUR (p, q) of SU(3) for the case $p \geqslant 2q$. The l values are plotted vertically and the j values of states, indicated by circles, are given near to the circles. Open arrows indicate the actions of the $A^{\pm\frac12}$, and full arrows indicate the actions of $\bar{A}^{\pm\frac12}$.

Theorem. For the state $|l, j\rangle$ of the IUR (p, q) we have

$$\bar{A}^{\frac12}|l,j\rangle = \left(\frac{(2l+1)(2l+j+1)(p-2l-j+1)(2l-q+j)}{2(l+1)}\right)^{\frac12}|l+\tfrac12, j\rangle \tag{41}$$

$$A^{\frac12}|l,j\rangle = (-1)^{j+1}\left(\frac{(j-1)(2l+1)(p-j+3)(q-j+2)}{2(l+1)}\right)^{\frac12}|l+\tfrac12, j-1\rangle \tag{42}$$

$$A^{-\frac12}|l,j\rangle = \left(\frac{(2l+1)(2l+j)(p-2l-j+2)(2l-q+j-1)}{2l}\right)^{\frac12}|l-\tfrac12, j\rangle \tag{43}$$

$$\bar{A}^{-\frac12}|l,j\rangle = (-1)^{j+1}\left(\frac{j(2l+1)(p-j+2)(q-j+1)}{2l}\right)^{\frac12}|l-\tfrac12, j+1\rangle \tag{44}$$

$$p_0|l,j\rangle = \tfrac{2}{3}(3l-q-p+3j-3)|l,j\rangle \tag{45}$$

$$\bar{A}^{-\frac12}A^{\frac12}|l,j\rangle = -(j-1)(p+3-j)(q+2-j)|l,j\rangle \tag{46}$$

$$A^{-\frac12}\bar{A}^{\frac12}|l,j\rangle = (p-2l+1-j)(2l+j+1)(2l-q+j)|l,j\rangle \tag{47}$$

$$\bar{A}^{\frac12}A^{-\frac12}|l,j\rangle = (p-2l+2-j)(2l+j)(2l-q+j-1)|l,j\rangle \tag{48}$$

$$A^{\frac12}\bar{A}^{-\frac12}|l,j\rangle = -j(p+2-j)(q+1-j)|l,j\rangle \tag{49}$$

$$\begin{aligned} Y_+|l,j\rangle &= A^{-\frac12}A^{\frac12}|l,j\rangle = A^{\frac12}A^{-\frac12}|l,j\rangle \\ &= (-1)^{j+1}[(j-1)(2l+j)(p-j+3)(p-2l-j+2)(q-j+2) \\ &\quad \times (2l-q-1+j)]^{\frac12}|l,j-1\rangle \end{aligned} \tag{50}$$

$$Y_-|l,j\rangle = -\bar{A}^{-\frac{1}{2}}\bar{A}^{\frac{1}{2}}|l,j\rangle = -\bar{A}^{\frac{1}{2}}\bar{A}^{-\frac{1}{2}}|l,j\rangle$$
$$= (-1)^l[j(2l+j+1)(p-j+2)(p-2l-j+1)(q-j+1)(2l-q+j)]^{\frac{1}{2}}|l,j+1\rangle. \tag{51}$$

In the case where $l=0$, (43) and (44) are undefined and should read

$$A^{-\frac{1}{2}}|0,q+1\rangle = \bar{A}^{-\frac{1}{2}}|0,q+1\rangle = 0.$$

Equations (41)–(44) imply the following range for j:

$$(a) \qquad p \geqslant 2q: \quad \begin{cases} q+1-2l \leqslant j \leqslant q+1, & 0 \leqslant l \leqslant \frac{1}{2}q \\ 1 \leqslant j \leqslant q+1, & \frac{1}{2}q \leqslant l \leqslant \frac{1}{2}(p-q) \\ 1 \leqslant j \leqslant p-2l+1, & \frac{1}{2}(p-q) \leqslant l \leqslant \frac{1}{2}p \end{cases}$$

$$(b) \qquad p \leqslant 2q: \quad \begin{cases} q+1-2l \leqslant j \leqslant q+1, & 0 \leqslant l \leqslant \frac{1}{2}(p-q) \\ q+1-2l \leqslant j \leqslant p-2l+1, & \frac{1}{2}(p-q) \leqslant l \leqslant \frac{1}{2}q \\ 1 \leqslant j \leqslant p-2l+1, & \frac{1}{2}q \leqslant l \leqslant \frac{1}{2}p. \end{cases}$$

The maximum degeneracy of l is $(q+1)$ for $p \geqslant 2q$, and $(p-q+1)$ for $p \leqslant 2q$.

Proof. We give a proof by induction, starting from $|\frac{1}{2}p, 1\rangle$ and working downwards.

Firstly, if one puts $l=\frac{1}{2}p, j=1$ in equations (41)–(51) one easily verifies that the actions of the various operators on $|\frac{1}{2}p, 1\rangle$ agree with those obtained earlier in this section.

Next, assume equations (41)–(51) to hold for all j in the appropriate range for some l' with $\frac{1}{2} \leqslant l' \leqslant \frac{1}{2}p$. We try to show this implies they hold for all j for $l = l' - \frac{1}{2}$. First note that

$$\langle l', j'|\bar{A}^{\frac{1}{2}}|l'-\tfrac{1}{2}, j\rangle \propto \langle l'-\tfrac{1}{2}, j|A^{-\frac{1}{2}}|l', j'\rangle^* = 0$$

unless $j=j'$, so $\bar{A}^{\frac{1}{2}}|l'-\frac{1}{2},j\rangle \propto |l',j\rangle$ and hence $A^{-\frac{1}{2}}\bar{A}^{\frac{1}{2}}|l'-\frac{1}{2},j\rangle \propto |l'-\frac{1}{2},j\rangle$. The action of $A^{-\frac{1}{2}}\bar{A}^{\frac{1}{2}}$ on $|l'-\frac{1}{2},j\rangle$ can then be obtained using

$$\langle l'-\tfrac{1}{2}, j|A^{-\frac{1}{2}}\bar{A}^{\frac{1}{2}}|l'-\tfrac{1}{2}, j\rangle = \langle l', j|\bar{A}^{\frac{1}{2}}A^{-\frac{1}{2}}|l', j\rangle.$$

In a similar manner one obtains the action of $\bar{A}^{-\frac{1}{2}}A^{\frac{1}{2}}$ on $|l'-\frac{1}{2},j\rangle$.

Next, using these actions together with equations (24), (26) and (27) we obtain the actions of p_0, $A^{\frac{1}{2}}\bar{A}^{-\frac{1}{2}}$ and $\bar{A}^{\frac{1}{2}}A^{-\frac{1}{2}}$ on $|l'-\frac{1}{2},j\rangle$. The action of $A^{\frac{1}{2}}$ and $\bar{A}^{\frac{1}{2}}$ on $|l'-\frac{1}{2},j\rangle$ can be found using equations (43), (44) and (49) and are found to be consistent with the formulae given in the theorem for $Y_+|l',j\rangle = A^{\frac{1}{2}}A^{-\frac{1}{2}}|l',j\rangle$ and $Y_-|l',j\rangle = -\bar{A}^{\frac{1}{2}}\bar{A}^{-\frac{1}{2}}|l',j\rangle$. The actions of $Y_+ = A^{-\frac{1}{2}}A^{\frac{1}{2}}$ and $Y_- = -\bar{A}^{-\frac{1}{2}}\bar{A}^{\frac{1}{2}}$ on $|l'-\frac{1}{2},j\rangle$ can also now be found.

Finally one defines $|l'-1,j\rangle$ to be a normalized state proportional to $A^{-\frac{1}{2}}|l'-\frac{1}{2},j\rangle$, its phase being determined by requiring $\langle l'-1,j|A^{-\frac{1}{2}}|l'-\frac{1}{2},j\rangle$ to be real and non-negative. This suffices to uniquely specify $|l'-\frac{1}{2},j\rangle$ except in the case, where $l \geqslant \frac{1}{2}(p-q)$, j has its maximum value $(p-2l+1)$. In that case $|l'-\frac{1}{2},j\rangle$ does not exist and we determine the phase of $|l'-1,j\rangle$ by the requirement that $\langle l'-1,j|\bar{A}^{-\frac{1}{2}}|l'-\frac{1}{2},j-1\rangle$ be real with sign equal to $(-1)^{j+1}$. The actions of $A^{-\frac{1}{2}}$ and $\bar{A}^{-\frac{1}{2}}$ on $|l'-\frac{1}{2},j\rangle$ are then determined using equations (16) and (17).

In all cases the formulae obtained are as stated in the theorem with l replaced by $l'-\frac{1}{2}$. The proof of the theorem then follows by induction.

From the results of the theorem, the actions of the $\overset{(-)}{O}{}^{\pm\frac{1}{2},\pm\frac{1}{2}}$ on states $|p,q;l,m,j\rangle$ can be obtained and thence, via the Wigner–Eckart theorem applied to SU(2), the matrix elements of the orginal $\overset{(-)}{q}_{\pm\frac{1}{2}}$. We shall not perform this perfectly straightforward calculation here.

4. Diagonalization of $(Y_+ + Y_-)$ and $i(Y_+ - Y_-)$

From equations (50) and (51) we see that $(Y_+ + Y_-)$ and $i(Y_+ - Y_-)$ have the following actions on $|l, j\rangle$:

$$(Y_+ + Y_-)|l, j\rangle = \beta_j|l, j-1\rangle + \beta_{(j+1)}|l, j+1\rangle \tag{52}$$

$$i(Y_+ - Y_-)|l, j\rangle = i\beta_j|l, j-1\rangle - i\beta_{(j+1)}|l, j+1\rangle \tag{53}$$

where

$$\beta_j = (-1)^{j+1}[(j-1)(2l+j)(p-j+3)(p-2l-j+2)(q-j+2)(2l-q+j-1)]^{\frac{1}{2}}. \tag{54}$$

From this the calculation of the eigenvalues and eigenvectors of the two operators can be performed. The matrix of coefficients for fixed l is tridiagonal and off-diagonal, and the characteristic equation for the eigenvalues involves continued fractions. The calculation of the eigenvalues for the top few l values results in increasingly complex expressions and does not lead the authors to believe that closed form expressions for the general case can be obtained easily. We content ourselves therefore with their calculation for particular examples.

The most trivial cases are the IUR $(p, 0)$ and (p, p), for which no degeneracies occur. In both cases all matrix elements of the two operators vanish as, therefore, do their eigenvalues. For the general IUR (p, q), the $l = p/2$ and $l = 0$ states also correspond to zero eigenvalues of $(Y_+ + Y_-)$ and $i(Y_+ - Y_-)$.

The simplest case for which non-zero values of the operators occur is the eight-dimensional (2, 1), for which $l = 1, \frac{1}{2}, 0$. $l = 1$ and $l = 0$ are non-degenerate and $l = \frac{1}{2}$ is doubly degenerate:

$$(Y_+ + Y_-)|1\rangle = i(Y_+ - Y_-)|1\rangle = 0$$

$$(Y_+ + Y_-)|\tfrac{1}{2}, 1\rangle = -3|\tfrac{1}{2}, 2\rangle,\ i(Y_+ - Y_-)|\tfrac{1}{2}, 1\rangle = 3i|\tfrac{1}{2}, 2\rangle$$

$$(Y_+ + Y_-)|\tfrac{1}{2}, 2\rangle = -3|\tfrac{1}{2}, 1\rangle,\ i(Y_+ - Y_-)|\tfrac{1}{2}, 2\rangle = -3i|\tfrac{1}{2}, 1\rangle$$

$$(Y_+ + Y_-)|0\rangle = i(Y_+ - Y_-)|0\rangle = 0.$$

The eigenvectors of $(Y_+ + Y_-)$ are easily found to be

$$|\tfrac{1}{2}, \pm I\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2}, 1\rangle \mp \frac{1}{\sqrt{2}}|\tfrac{1}{2}, 2\rangle \tag{55}$$

where

$$(Y_+ + Y_-)|\tfrac{1}{2}, \pm I\rangle = \pm 3|\tfrac{1}{2}, \pm I\rangle. \tag{56}$$

For $i(Y_+ - Y_-)$ we have

$$|\tfrac{1}{2}, \pm I'\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2}, 1\rangle \mp \frac{i}{\sqrt{2}}|\tfrac{1}{2}, 2\rangle \tag{57}$$

where

$$i(Y_+ - Y_-)|\tfrac{1}{2}, \pm I'\rangle = \pm 3|\tfrac{1}{2}, \pm I'\rangle. \tag{58}$$

For the 27-dimensional IUR (4, 2), $l=\frac{3}{2},\frac{1}{2}$ are doubly degenerate and $l=1$ is triply degenerate. The eigenvectors of (Y_++Y_-) are

$$|\tfrac{3}{2},\pm I\rangle=\frac{1}{\sqrt{2}}|\tfrac{3}{2},1\rangle\mp\frac{1}{\sqrt{2}}|\tfrac{3}{2},2\rangle$$

where

$$(Y_++Y_-)|\tfrac{3}{2},\pm I\rangle=\pm 10|\tfrac{3}{2},\pm I\rangle,$$

$$|1,\pm I\rangle=\tfrac{1}{2}|1,1\rangle\mp\frac{1}{\sqrt{2}}|1,2\rangle-\tfrac{1}{2}|1,3\rangle$$

$$|1,0\rangle=\frac{1}{\sqrt{2}}|1,1\rangle+\frac{1}{\sqrt{2}}|1,3\rangle$$

where

$$(Y_++Y_-)|1,\pm I\rangle=\pm 4\sqrt{10}|1,\pm I\rangle,\ (Y_++Y_-)|1,0\rangle=0,$$

and

$$|\tfrac{1}{2},\pm I\rangle=\frac{1}{\sqrt{2}}|\tfrac{1}{2},2\rangle\pm\frac{1}{\sqrt{2}}|\tfrac{1}{2},3\rangle$$

where

$$(Y_++Y_-)|\tfrac{1}{2},\pm I\rangle=\pm 8|\tfrac{1}{2},\pm I\rangle.$$

For $\mathrm{i}(Y_+-Y_-)$ the eigenvectors are

$$|\tfrac{3}{2},\pm I'\rangle=\frac{1}{\sqrt{2}}|\tfrac{3}{2},1\rangle\mp\frac{\mathrm{i}}{\sqrt{2}}|\tfrac{3}{2},2\rangle$$

$$|1,\pm I'\rangle=\tfrac{1}{2}|1,1\rangle\pm\frac{\mathrm{i}}{\sqrt{2}}|1,2\rangle-\tfrac{1}{2}|1,3\rangle$$

$$|1,0'\rangle=\frac{1}{\sqrt{2}}|1,1\rangle-\frac{1}{\sqrt{2}}|1,3\rangle$$

$$|\tfrac{1}{2},\pm I'\rangle=\frac{1}{\sqrt{2}}|\tfrac{1}{2},2\rangle\mp\frac{\mathrm{i}}{\sqrt{2}}|\tfrac{1}{2},3\rangle.$$

The primed states correspond to the same eigenvalues of $\mathrm{i}(Y_+-Y_-)$ as the corresponding unprimed states did to (Y_++Y_-).

5. The pseudoscalar meson octet

We consider in this section the SU(3) symmetry of strong interactions of hadrons, specifically the pseudoscalar mesons which transform as the (2, 1) IUR. If we interpret L^2, l_0 and p_0 as (total isotopic spin)^2, the third component of isotopic spin, and hypercharge, respectively, the particles may be identified with the $|l,m;j\rangle$ states as follows:

$$\begin{array}{lll}\pi^\pm\equiv|1,\pm1;1\rangle, & \pi^0\equiv|1,0;1\rangle, & \eta\equiv|0,0;1\rangle\\ \bar{\mathrm{K}}^0\equiv|\tfrac{1}{2},\tfrac{1}{2};1\rangle, & \mathrm{K}^-\equiv|\tfrac{1}{2},-\tfrac{1}{2};1\rangle & \\ \mathrm{K}^+\equiv|\tfrac{1}{2},\tfrac{1}{2};2\rangle, & \mathrm{K}^0\equiv|\tfrac{1}{2},-\tfrac{1}{2};2\rangle. & \end{array}\tag{59}$$

However, once weak interactions are admitted, K^0 and $\bar{K}^0$ are unstable and the states which have an exponential time dependence are not K^0 and $\bar{K}^0$ but a linear combination of them, one short-lived K_S^0, the other K_L^0 with a lifetime of order 10^3 longer than K_S^0 (see Steinberger 1970). These are no longer eigenstates of hypercharge, and we show that they correspond to the diagonalization of a modification of the $(Y_+ + Y_-)$, $i(Y_+ - Y_-)$ operators considered in the previous sections. We divide our considerations into three cases: (*a*) assuming CP is conserved (which is now known not to be the case); (*b*) assuming CP is not conserved but CPT is conserved (thought to be the true situation); and (*c*) assuming T is conserved, but not CP. The reader is referred to the article by Steinberger (1970) for a detailed discussion of CP violation and the K^0 particles, and for a derivation of the equations quoted here for the three cases.

(*a*) *CP conserved.* In this case we have

$$K_S^0 = \frac{1}{\sqrt{2}} K^0 + \frac{1}{\sqrt{2}} \bar{K}^0, \qquad K_L^0 = \frac{1}{\sqrt{2}} K^0 - \frac{1}{\sqrt{2}} \bar{K}^0$$

or

$$K_S^0 = \frac{1}{\sqrt{2}} |\tfrac{1}{2}, -\tfrac{1}{2}; 2\rangle + \frac{1}{\sqrt{2}} |\tfrac{1}{2}, \tfrac{1}{2}; 1\rangle \tag{60}$$

$$K_L^0 = \frac{1}{\sqrt{2}} |\tfrac{1}{2}, -\tfrac{1}{2}; 2\rangle - \frac{1}{\sqrt{2}} |\tfrac{1}{2}, \tfrac{1}{2}; 1\rangle. \tag{61}$$

These look rather like the eigenstates of $(Y_+ + Y_-)$ given in equation (55) except that the m-values are wrong. The actual eigenstates of $(Y_+ + Y_-)$ are $(\bar{K}^0 \pm K^+)/\sqrt{2}$ and $(K^0 \pm K^-)/\sqrt{2}$, which are non-physical since they mix charge. However, if we introduce operators

$$A = Y_+ l_+ + Y_- l_-, \qquad B = i(Y_+ l_+ - Y_- l_-) \tag{62}$$

both of which are also Hermitian, and use the easily derived results

$$l_- \bar{K}^0 = K^-, \qquad l_- K^+ = K^0, \qquad l_+ K^0 = K^+, \qquad l_+ K^- = \bar{K}^0$$

we find that

$$A K_S^0 = -3 K_S^0, \qquad A K_L^0 = 3 K_L^0. \tag{63}$$

Furthermore the mesons $\pi^\pm$, π^0, η and $K^\pm$ are also eigenstates of A corresponding to the eigenvalue 0. Now A, in addition to commuting with $(\text{isospin})^2 = L^2$, also commutes with the charge operator $Q = (l_0 + \tfrac{1}{2} p_0)$. We see, therefore, that for the psuedoscalar meson octet it is the operators (L^2, Q, A) that are diagonalized rather than the usual (L^2, l_0, p_0).

(*b*) *CPT conserved, CP non-conserved.* In this case we have

$$K_S^0 = [2(1 + |\epsilon|^2)]^{-\frac{1}{2}}((1+\epsilon)K^0 + (1-\epsilon)\bar{K}^0) \tag{64}$$

$$K_L^0 = [2(1 + |\epsilon|^2)]^{-\frac{1}{2}}((1+\epsilon)K^0 - (1-\epsilon)\bar{K}^0) \tag{65}$$

where ϵ is a complex parameter determined by the extent of CP violation.

One may fairly easily show that if one introduces the operator

$$G = A + 2i\epsilon(1+\epsilon^2)^{-1}B \tag{66}$$

then

$$GK_S^0 = \frac{-3(1-\epsilon^2)}{1+\epsilon^2}K_S^0, \qquad GK_L^0 = \frac{3(1-\epsilon^2)}{1+\epsilon^2}K_L^0. \tag{67}$$

The other pseudoscalar mesons are also eigenstate of G, corresponding to the eigenvalue 0. Hence in this case, since G also commutes with Q, it is the operators (L^2, Q, G) which are diagonal for the physically realized pseudoscalar mesons. Note that, due to the complexity of ϵ, G is not a Hermitian operator; correspondingly its eigenvalues are not all real, and K_S^0 and K_L^0 are not orthogonal states.

(*c*) *T conserved, CP non-conserved.* In this final case

$$K_S^0 = [2(1+|\epsilon|^2)]^{-\frac{1}{2}}((1+\epsilon)K^0 + (1-\epsilon)\bar{K}^0) \tag{68}$$

$$K_L^0 = [2(1+|\epsilon|^2)]^{-\frac{1}{2}}((1-\epsilon)K^0 - (1+\epsilon)\bar{K}^0). \tag{69}$$

Now one easily finds that no operator of the form $(A+\alpha B)$ exists, of which both K_S^0 and K_L^0 are eigenstates. Instead one finds that

$$(A + 2i\epsilon(1+\epsilon^2)^{-1}B)K_S^0 = -3(1-\epsilon^2)(1+\epsilon^2)^{-1}K_S^0 \tag{70}$$

and

$$(A - 2i\epsilon(1+\epsilon^2)^{-1}B)K_L^0 = 3(1-\epsilon^2)(1+\epsilon^2)^{-1}K_L^0. \tag{71}$$

Hence it is not clear in this case whether three operators may be found which are all diagonal for the pseudoscalar mesons.

Case (*b*) is the one which is now believed to be the correct description. The authors have been unable to find a physical interpretation of the operators A, B and G. The most that appears possible to state is that (L^2, Q, G) is more appropriate a set of state labelling operators for the pseudoscalar meson octet than (L^2, l_0, p_0) in that the physically realized particles are eigenstates of the former rather than the latter set of operators. One could conjecture that the same applies to the vector meson octet, although the value of ϵ appearing in the definitions of G might well be different. However, the lifetimes of the K^* resonances are so short that it is not experimentally possible to detect whether there are distinct particles K_L^{*0} and K_S^{*0} with different lifetimes, let alone whether any CP violation occurs.

(L^2, l_0, p_0) is definitely more appropriate for the baryon and anti-baryon octets, since the neutron is an eigenstate of p_0, not of A, B or G. For the decuplet or quark triplets, no l-degeneracy occurs and so all eigenstates of (L^2, l_0, p_0) correspond to zero eigenvalues of $Y_\pm$, and are therefore automatically also eigenstates of (L^2, Q, G). Also, the fact that all the pseudoscalar mesons other than K_L^0 and K_S^0 correspond to zero eigenvalues of G makes it difficult to put a physical scale to G. Only if particles were found which fit into the 27-dimensional IUR would it be possible to obtain further information as to the possible physical interpretation of G.

Finally, note that some, but not all, $\Delta S \neq 0$ decays of pseudoscalar mesons conserve G. Those of the pseudoscalar mesons other than the K^0's certainly do, since the G-value of all particles concerned is zero. The K_L^0 and K_S^0 decays clearly do not conserve G.

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